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Drag modifications for a sphere in a rotational motion at small, non-zero Reynolds and Taylor numbers: wake interference and possibly Coriolis effects

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Matched asymptotic expansion methods are used to establish governing equations of Oseen type for a tethered sphere that describes a circular path and a stationary sphere subjected to a rotating fluid in an 'antisedimentation' tube. The two cases are shown to be significantly different, in contrast to an earlier presentation (Davis & Brenner 1986), because only the latter is subject to the Coriolis force. The evaluation of the force and torque coefficients is much improved, enabling better comparisons to be made with the classical rectilinear trajectory result of Proudman & Pearson (1957).

1. Introduction

This paper seeks to correct an error and improve the analysis in an earlier paper (Davis & Brenner 1986, referred to hereinafter as I), that principally addressed the steady rotation of a tethered sphere through a viscous fluid at small, non-zero Taylor and Reynolds numbers, the Rossby number being assumed to be of order unity. Since the sphere-centre trajectory is closed upon itself, the sphere moves continuously through the residual disturbance left by its own wake and hence the drag that it experiences is likely to differ from that appropriate to the same translational Reynolds number in rectilinear instead of curvilinear motion. Such differences are necessarily absent from the quasi-static Stokes limit and only become possible when fluid inertial and/or Coriolis and/or centrifugal effects are sensible. It will be shown that the 'antisedimentation' rotating tube device of Dill & Brenner (1983) and Nadim, Cox & Brenner (1985), presented in I as being essentially described by the tethered sphere analysis, differs significantly from the latter in being subject to the Coriolis force which plays no role in the correct governing equations for the fluid motion generated by the tethered sphere.

After first describing why the Coriolis force enters one calculation but not the other, numerical values are given for the non-Stokesian contributions to the hydrodynamic force and torque, thus enabling the drag results to be compared with the rectilinear Oseen (1927) and Proudman & Pearson (1957) Reynolds number correction to Stokes law. These evaluations are based on the leading order, outer field solutions for the two problems, obtained by the use of matched asymptotic expansions. The details of the inner fields, being the same as I in each case, are omitted so each analysis in the subsequent sections starts with an Oseen linearization. The corrected versions of the analysis in I are significantly improved by expressing the Fourier sums as integrals over a semi-infinite range instead of one period and thus enabling the triple integrals, similar to those computed in I, to be reduced to single or double integrals with simpler integrands. Formulae quoted without reference may be found in Gradshteyn & Ryzhik (1980).

2. Basic equations and discussion

In general, the equations of motion of an incompressible Newtonian fluid from the viewpoint of an observer fixed in a steadily rotating system with angular velocity $\boldsymbol{\Omega}$ are (Greenspan 1968)

$$\frac{\partial \boldsymbol{v}'}{\partial t'}+\boldsymbol{v}'\cdot\boldsymbol{\nabla}'\boldsymbol{v}'+2\boldsymbol{\Omega}\times\boldsymbol{v}' = -\frac{1}{\rho^*}\boldsymbol{\nabla}'p'+\nu\nabla'^2\boldsymbol{v}', \quad (2.1)$$

$$\boldsymbol{\nabla}'\cdot\boldsymbol{v}' = 0, \quad (2.2)$$

with

$$p' = \pi'-\tfrac{1}{2}\rho^*(\boldsymbol{\Omega}\times\boldsymbol{r}')^2, \quad (2.3)$$

in which the true fluid-mechanical pressure π' is augmented by the fictitious fluid pressure due to the centrifugal force on the fluid. Here ν and ρ^* denote the kinematic viscosity and density respectively and $\boldsymbol{r}'$ is the position vector of a fluid point measured from an origin lying on the axis of rotation which is taken to be the z-axis, i.e. $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}$. The external body force due to gravity has been absorbed into p'.

Thus the lengthscale $(\nu/\Omega)^{\frac{1}{2}}$ and timescale Ω^{-1} are given so a velocity scale U enables a Reynolds number to be defined by

$$Re = U/(\nu\Omega)^{\frac{1}{2}}. \quad (2.4)$$

Then, on introducing dimensionless variables by setting

$$\boldsymbol{r}' = \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\boldsymbol{r},\quad \boldsymbol{\nabla}' = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\boldsymbol{\nabla},\quad t' = \Omega^{-1}t,\quad \boldsymbol{v}' = U\boldsymbol{v},\quad p'-p'_\infty = \rho^*U(\Omega\nu)^{\frac{1}{2}}P, \quad (2.5)$$

the governing equations (2.1)–(2.3) become

$$\frac{\partial \boldsymbol{v}}{\partial t}+Re\,(\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}+2\hat{\boldsymbol{z}}\times\boldsymbol{v} = -\boldsymbol{\nabla}P+\nabla^2\boldsymbol{v}, \quad (2.6)$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \quad (2.7)$$

$$P = \Pi-\tfrac{1}{2}(Re)^{-1}\,(\hat{\boldsymbol{z}}\times\boldsymbol{r})^2. \quad (2.8)$$

Now, if the rigid body rotation corresponding to the rotating axes is subtracted out by writing

$$\boldsymbol{v} = \boldsymbol{q}-(Re)^{-1}\,\hat{\boldsymbol{z}}\times\boldsymbol{r}, \quad (2.9)$$

then $\boldsymbol{\nabla}\cdot\boldsymbol{q} = 0$, the pressure/density ratio is Π and

$$\frac{\partial \boldsymbol{q}}{\partial t}+Re\,(\boldsymbol{q}\cdot\boldsymbol{\nabla})\,\boldsymbol{q}-(\hat{\boldsymbol{z}}\times\boldsymbol{r})\cdot\boldsymbol{\nabla}\boldsymbol{q}+\hat{\boldsymbol{z}}\times\boldsymbol{q} = -\boldsymbol{\nabla}\Pi+\nabla^2\boldsymbol{q}, \quad (2.10)$$

i.e. on introducing cylindrical polar coordinates (ρ,ϕ,z),

$$\frac{\partial \boldsymbol{q}}{\partial t}+Re\,(\boldsymbol{q}\cdot\boldsymbol{\nabla})\,\boldsymbol{q}-\left[\frac{\partial q_\rho}{\partial\phi}\hat{\boldsymbol{\rho}}+\frac{\partial q_\phi}{\partial\phi}\hat{\boldsymbol{\phi}}+\frac{\partial q_z}{\partial\phi}\hat{\boldsymbol{z}}\right] = -\boldsymbol{\nabla}\Pi+\nabla^2\boldsymbol{q}. \quad (2.11)$$

Similarly, if in the equations of motion referred to fixed axes, namely (2.7) and

$$\frac{\partial \boldsymbol{v}}{\partial t}+Re\,(\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} = -\boldsymbol{\nabla}\Pi+\nabla^2\boldsymbol{v}, \quad (2.12)$$

a positive rigid-body rotation is subtracted out by writing

$$v = q + (Re)^{-1}\,\hat{z}\times r, \tag{2.13}$$

then $\nabla\cdot q = 0$ and

$$\frac{\partial q}{\partial t} + Re\,(q\cdot\nabla)\,q + 2\hat{z}\times q + \left[\frac{\partial q_\rho}{\partial\phi}\hat{\rho} + \frac{\partial q_\phi}{\partial\phi}\hat{\phi} + \frac{\partial q_z}{\partial\phi}\hat{z}\right] = -\nabla P + \nabla^2 q, \tag{2.14}$$

where P is related to Π by (2.8). Here the time derivatives are computed by holding the spatial coordinates fixed in either frame of reference.

If the fluid is at rest in a fixed frame at infinity or at a fixed cylindrical boundary and the rotation is due to some forcing mechanism such as the tethered sphere describing a circular trajectory, then the unperturbed flow is given by $v = 0 = \nabla\Pi$ in (2.12) or $v = -(Re)^{-1}\hat{z}\times r$, $\nabla P = 0$ in (2.6), i.e. $q = 0 = \nabla\Pi$ in (2.11) and equivalent equations for the perturbation flow are obtained in the Stokes limit by seeking solutions of the form $q \equiv q(\rho,\phi,z)$ in (2.11) or $v \equiv v(\rho,\phi-t,z)$ in (2.12). Alternatively, if the rotation is applied at infinity or at a fixed cylindrical boundary such as in the anti-sedimentation tube, then the unperturbed flow is given by $v = 0 = \nabla P$ in (2.6) or $v = (Re)^{-1}\hat{z}\times r$ in (2.12), i.e. $q = 0 = \nabla P$ in (2.14) and equivalent equations for the perturbation flow are obtained in the Stokes limit by seeking solutions of the form $q \equiv q(\rho,\phi,z)$ in (2.14) or $v \equiv v(\rho,\phi+t,z)$ in (2.6).

The cases are distinguished by the presence or absence of the Coriolis force according to whether there is a rotation in the far field. Their fundamental difference can be readily illustrated by means of the velocity representation

$$q = \nabla\times\nabla\times(\hat{z}f) + \nabla\times(\hat{z}g),$$

which eventually yields, in the two cases

$$\left[\nabla^2 - \frac{\partial}{\partial\phi}\right]^2 \nabla^2 f + 4\frac{\partial^2 f}{\partial z^2} = 0, \tag{2.15}$$

or

$$\left[\nabla^2 + \frac{\partial}{\partial\phi}\right]\nabla^2 f = 0 = \left[\nabla^2 + \frac{\partial}{\partial\phi}\right] g. \tag{2.16}$$

Thus the sixth-order system exhibited by the characteristic polynomial in (4.5) decouples, in the absence of the Coriolis term, into two systems of fourth and second order. It should be noted that, though (2.15) fortuitously retains constant coefficients after introduction of the azimuthal convection, the solutions are considerably more complicated than for the sixth-order system involving only rotation (Herron, Davis & Bretherton 1975; Smith 1981) or even rotation and linear convection (Childress 1964).

As in I, $T^{\frac{1}{2}}$ and B are defined to be the dimensionless forms of the sphere radius a and the 'path' radius b ($\gg a$), namely, from (2.5),

$$T^{\frac{1}{2}} = a\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}, \quad B = b\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}, \tag{2.17}$$

with B assumed to be $O(1)$ and hence $T^{\frac{1}{2}} \ll 1$. The Taylor number is related to the Reynolds number Re in (2.4) by $Re = T^{\frac{1}{2}}B$. The dimensionless force F and torque $M\hat{z}$ exerted on the sphere by the fluid are given by

$$\left.\begin{aligned} F &= -6\pi[(1+T^{\frac{1}{2}}C_y)\,\hat{y} - T^{\frac{1}{2}}C_x\,\hat{x} + O(T)], \\ M &= -8\pi[1 - T^{\frac{1}{2}}C_z + O(T)], \end{aligned}\right\} \tag{2.18}$$

B	C_x	C_y	C_z	C
(0)	$1/\sqrt{2}$	$1/\sqrt{2}$	0	∞
0.5	0.6849	0.7193	0.0848	1.4386
1.0	0.6255	0.7596	0.1510	0.7596
1.5	0.5461	0.8355	0.1901	0.5570
2.0	0.4650	0.9506	0.2059	0.4753
2.5	0.3937	1.1010	0.2081	0.4404
3.0	0.3361	1.2778	0.2051	0.4259
3.5	0.2914	1.4712	0.2016	0.4203
4.0	0.2566	1.6739	0.1988	0.4185
4.5	0.2291	1.8816	0.1969	0.4181
5.0	0.2069	2.0919	0.1956	0.4184

TABLE 1. Numerical values, for various radii of order unity, of the force, torque and normalized Reynolds-number drag coefficients for the tethered sphere. (The extreme value $B = 0$ is inadmissible but is included here only for completeness.)

B	C_x	C_y	C_z	C
(0)	0.0517	0.5241	0	∞
0.5	0.0630	0.5485	0.0072	1.0970
1.0	0.0747	0.6193	0.0108	0.6193
1.5	0.0873	0.7293	0.0099	0.4862
2.0	0.0942	0.8680	0.0065	0.4340
2.5	0.0948	1.0248	0.0028	0.4099
3.0	0.0905	1.1910	−0.0003	0.3970
3.5	0.0839	1.3617	−0.0027	0.3891
4.0	0.0767	1.5308	−0.0047	0.3827
4.5	0.0702	1.7085	−0.0066	0.3797
5.0	0.0646	1.8836	−0.0086	0.3767

TABLE 2. Numerical values, for various radii of order unity, of the force, torque and normalized Reynolds-number drag coefficients for the sphere in an anti-sedimentation tube. (The extreme value $B = 0$ is inadmissible but is included here only for completeness.)

where the dependence of the coefficients C_x, C_y and C_z on B is derived in the subsequent sections.

Though the two cases have the same relative motion of sphere and fluid, they are distinguished by the appearance of the Coriolis force when there is a rotation in the far field, as explained above, and by the different formulae for the force and torque coefficients in (3.17) and (4.10). The results displayed in Tables 1 and 2 exhibit few similarities, the most notable being the behaviour of $C = C_y/B$ as B increases. Since the dimensionless force exerted by the sphere on the fluid in the direction of relative motion is, from (2.18), $6\pi(1+C\,Re)$, the values of C enable comparison to be made with the classical Oseen (1927) and Proudman & Pearson (1957) rectilinear Stokes-law drag correction coefficient. As remarked in I, the inner field equations become identical in the $B \to \infty$ limit but the same is not true of the outer equations discussed here because the convective terms in (3.1) and (4.1) and the Coriolis term in (4.1) remain, despite the Taylor number being identically zero. Indeed, the $B \to \infty$ limit is inadmissible in this analysis, since only B-values of order unity are consistent with this scaling hypothesis $Re = T^{\frac{1}{2}}B$ and choice of origin. Thus it is of interest that the displayed values of C, whose accuracy worsens as B increases, become very close or moderately close to $\frac{3}{8}$ for the anti-sedimentation tube and tethered sphere respectively. Though the same convective terms and half the Coriolis force were retained in I, exact calculations for the $B = 0$ case indicate that the results obtained in I had possibly good reason to be unrelated to the 'averages' of those listed here in tables 1 and 2.

On the other hand, significant improvement is achieved in the subsequent analysis by reducing the triple integrals to single or double integrals with the aid of a rearrangement that is analogous to the formula for the Laplace transform of a periodic function. In physical terms, the expressions (3.10), (3.16), (4.8) and (4.9) exhibit the relevance of the entire history of previous rotations in determining the wake interference effects.

3. The tethered sphere

A sphere of radius a describes a path of radius b ($\gg a$) at uniform speed $U = \Omega b$. Since the inner field was considered adequately in I, it suffices here to note that the outer field sees the sphere's motion as a point force of magnitude $6\pi\rho^*\nu Ua$ which, after non-dimensionalization, is expressed by including the forcing term $6\pi T^{\frac{1}{2}}\hat{\boldsymbol{y}}\delta(x-B)\,\delta(y)\,\delta(z)$ on the right-hand side of (2.11). Thus, on writing $\boldsymbol{q} \sim T^{\frac{1}{2}}\boldsymbol{Q}(\boldsymbol{r})$, $\Pi \sim T^{\frac{1}{2}}\Pi_0(\boldsymbol{r})$ and including the point force term in (2.11), the perturbation field is, to leading order, governed by

$$\nabla^2\boldsymbol{Q}-\nabla\Pi_0+\frac{\partial Q_\rho}{\partial\phi}\hat{\boldsymbol{\rho}}+\frac{\partial Q_\phi}{\partial\phi}\hat{\boldsymbol{\phi}}+\frac{\partial Q_z}{\partial\phi}\hat{\boldsymbol{z}} = -6\pi\hat{\boldsymbol{y}}\,\delta(\boldsymbol{r}-\hat{\boldsymbol{x}}B), \tag{3.1}$$

$$\nabla\cdot\boldsymbol{Q} = 0, \tag{3.2}$$

$$\boldsymbol{Q}\to\boldsymbol{0}\quad \text{as}\ |\boldsymbol{r}|\to\infty. \tag{3.3}$$

The important error in I was to retain in (3.1) the term $-\hat{\boldsymbol{z}}\times\boldsymbol{Q}$ which should be cancelled by the ϕ-derivatives of the unit vectors $\hat{\boldsymbol{\rho}},\hat{\boldsymbol{\phi}}$ in $\partial\boldsymbol{Q}/\partial\phi$ (see (2.10), (2.11)). When appropriate changes are made to the analysis in §4.2 of I, the correct, simplified expressions for the velocity components satisfying (3.1)–(3.3) and the z-component of vorticity are found to be

$$Q_z = -\frac{3\mathrm{i}}{2\pi}\int_0^\infty \lambda\sin\lambda z\,\mathrm{d}\lambda\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m\phi}\int_0^\infty\frac{\xi^2J_m(\rho\xi)}{(\lambda^2+\xi^2)(\lambda^2+\xi^2-\mathrm{i}m)}[J_{m-1}(B\xi)+J_{m+1}(B\xi)]\,\mathrm{d}\xi, \tag{3.4}$$

$$Q_\rho+\mathrm{i}Q_\phi = -\frac{3\mathrm{i}}{2\pi}\int_0^\infty\cos\lambda z\,\mathrm{d}\lambda\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m\phi}\int_0^\infty\frac{\xi J_{m+1}(\rho\xi)}{(\lambda^2+\xi^2-\mathrm{i}m)}\times\left\{-2J_{m+1}(B\xi)+\frac{\xi^2}{\lambda^2+\xi^2}[J_{m-1}(B\xi)+J_{m+1}(B\xi)]\right\}\mathrm{d}\xi, \tag{3.5}$$

$$\omega_z = -\frac{3}{2\pi}\int_0^\infty\cos\lambda z\,\mathrm{d}\lambda\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m\phi}\int_0^\infty\frac{\xi^2J_m(\rho\xi)}{\lambda^2+\xi^2-\mathrm{i}m}[J_{m-1}(B\xi)-J_{m+1}(B\xi)]\,\mathrm{d}\xi. \tag{3.6}$$

The denominators are indicative of the second- and fourth-order equations (2.16).

Now, if the periodic functions $s(\phi)$, $S(\phi)$ are defined by

$$s(\phi) = \sum_{m=-\infty}^{\infty}s_m\,\mathrm{e}^{\mathrm{i}m\phi},\quad S(\phi) = \sum_{m=-\infty}^{\infty}\frac{s_m\,\mathrm{e}^{\mathrm{i}m\phi}}{\lambda^2+\xi^2-\mathrm{i}m}, \tag{3.7}$$

then it may readily be shown that

$$S(\phi) = \int_\phi^\infty s(\psi)\exp[-(\lambda^2+\xi^2)(\psi-\phi)]\,d\psi. \tag{3.8}$$

The summations $s(\phi)$ that appear in (3.4)–(3.6) have closed form expressions listed in the Appendix and, when (3.8) is applied to (3.6), the λ and ξ integrals can be evaluated exactly to yield

$$\omega_z = \frac{3}{8\pi^{\frac{1}{2}}}\int_\phi^\infty \frac{B-\rho\cos\psi}{(\psi-\phi)^{\frac{5}{2}}}\exp\left\{-\frac{z^2+\rho^2+B^2-2\rho B\cos\psi}{4(\psi-\phi)}\right\}d\psi. \tag{3.9}$$

In particular, the rotation induced at the point singularity which represents the sphere is given by

$$\tfrac{1}{2}(\omega_z)_{(B,0,0)} = \frac{3}{8\pi^{\frac{1}{2}}}\int_0^\infty \frac{\sin^2\frac{1}{2}\psi}{\psi^{\frac{5}{2}}}\exp\left\{-\frac{B^2}{\psi}\sin^2\tfrac{1}{2}\psi\right\}d\psi, \tag{3.10}$$

which, when substituted in

$$M = -8\pi[1-\tfrac{1}{2}T^{\frac{1}{2}}(\omega_z)_{B,0,0)}+O(T)], \tag{3.11}$$

determines the torque coefficient C_z in (2.18).

The dimensionless force is given similarly by (Herron *et al.* 1975)

$$\boldsymbol{F} = -6\pi[\hat{\boldsymbol{y}}-T^{\frac{1}{2}}\{(Q_\rho-Q_\rho^{(s)})_{(B,0,0)}\hat{\boldsymbol{x}}+(Q_\phi-Q_\phi^{(s)})_{(B,0,0)}\hat{\boldsymbol{y}}\}+O(T)], \tag{3.12}$$

where $\boldsymbol{Q}^{(s)}$ denotes the Stokes solution which must be subtracted out in order to determine the reflected velocity due to the inertia terms in (3.1). Thus the summations required in (3.5) are, in the notation of (3.7), of the form

$$S(0)-\frac{s(0)}{\lambda^2+\xi^2} = \int_0^\infty [s(\psi)-s(0)]\exp[-(\lambda^2+\xi^2)\psi]\,d\psi, \tag{3.13}$$

and hence

$$\begin{aligned}
&(Q_\rho+iQ_\phi-Q_\rho^{(s)}-iQ_\phi^{(s)})_{(B,0,0)} \\
&= \frac{3i}{2\pi}\int_0^\infty d\lambda\int_0^\infty \xi\,d\xi\int_0^\infty \exp[-(\lambda^2+\xi^2)\psi] \\
&\quad\times\left\{\left[2-\frac{\xi^2}{\lambda^2+\xi^2}\right]\left[J_0(2B\xi\sin\tfrac{1}{2}\psi)\exp(-i\psi)-1\right]+\frac{\xi^2}{\lambda^2+\xi^2}J_2(2B\xi\sin\tfrac{1}{2}\psi)\right\}d\psi \\
&= -\frac{3i}{4\pi^{\frac{1}{2}}}\int_0^\infty\left[1-\exp\left(-i\psi-\frac{B^2}{\psi}\sin^2\tfrac{1}{2}\psi\right)\right]\psi^{-\frac{3}{2}}\,d\psi \\
&\quad+\frac{3i}{4\pi}\int_0^\infty\int_0^1 \chi^{\frac{1}{2}}(1-\eta^2)\,d\chi\,d\eta\int_0^\infty \exp(-\chi\psi)\{J_2[2B\chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\sin\tfrac{1}{2}\psi] \\
&\quad-J_0[2B\chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\sin\tfrac{1}{2}\psi]\exp(-i\psi)+1\}\,d\psi,
\end{aligned} \tag{3.14}$$

where the 'polar' coordinates χ, η are defined by

$$\lambda = \chi^{\frac{1}{2}}\eta, \quad \xi = \chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}. \tag{3.15}$$

In the first term of (3.14), the ξ-integration has been evaluated, as in (3.9), by using the formula

$$\int_0^\infty \exp(-\alpha\xi^2)\,J_\nu(\beta\xi)\,\xi^{\nu+1}\,d\xi = \frac{1}{2\alpha}\left(\frac{\beta}{2\alpha}\right)^\nu \exp\left(-\frac{\beta^2}{4\alpha}\right).$$

In the second term of (3.14), the η-integrations can be derived from the result

$$\int_0^1 J_0[\alpha(1-\eta^2)^{\frac{1}{2}}]\,d\eta = \frac{\sin\alpha}{\alpha},$$

after which the χ-integrations are elementary. Thus (3.14) can be reduced to the form

$$(Q_\rho - Q_\rho^{(s)})_{(B,0,0)} + i(Q_\phi - Q_\phi^{(s)})_{(B,0,0)}$$
$$= \frac{3i}{4\pi^{\frac{1}{2}}}\int_0^\infty \left\{\frac{(3+\exp(-i\psi))}{4B^2\sin^2\frac{1}{2}\psi}\left[\frac{\pi^{\frac{1}{2}}}{2B\sin\frac{1}{2}\psi}\operatorname{erf}\left(\frac{B\sin\frac{1}{2}\psi}{\psi^{\frac{1}{2}}}\right) - \psi^{-\frac{1}{2}}\exp\left(-\frac{B^2}{\psi}\sin^2\tfrac{1}{2}\psi\right)\right]\right.$$
$$\left. -\psi^{-\frac{3}{2}}\left[\tfrac{2}{3}+\tfrac{1}{2}(1-\exp[-i\psi])\exp\left(-\frac{B^2}{\psi}\sin^2\tfrac{1}{2}\psi\right)\right]\right\}d\psi, \quad (3.16)$$

where the error function is defined by

$$\operatorname{erf}(x) = \frac{2}{\pi^{\frac{1}{2}}}\int_0^x \exp(-y^2)\,dy.$$

Values of the integrals in (3.10) and (3.16) have been computed numerically in order to tabulate in table 1 the B-dependent force and torque coefficients in the formulae (2.18) where (cf. (3.11), (3.12))

$$C_x = (Q_\rho - Q_\rho^{(s)})_{(B,0,0)}, \quad C_y = (Q_\phi - Q_\phi^{(s)})_{(B,0,0)}, \quad C_z = \tfrac{1}{2}(\omega_z)_{(B,0,0)}. \quad (3.17)$$

The additional reflected velocity due to the presence of a fixed cylindrical wall at $\rho = D$ ($> B$) involves, for each $m \neq 0$, Bessel functions with complex arguments. The dominant contribution, for $D \gg B$, arises from the swirling motion given by the $m = 0$ terms in (3.4) and (3.5), namely

$$Q_{0z} = 0 = Q_{0\rho}, \quad Q_{0\phi} = \frac{3}{\pi}\int_0^\infty \cos\lambda z\,d\lambda \int_0^\infty \frac{\xi J_1(\rho\xi)}{\lambda^2+\xi^2} J_1(B\xi)\,d\xi$$
$$= \frac{3}{\pi}\int_0^\infty I_1(B\lambda)K_1(\rho\lambda)\cos\lambda z\,d\lambda \quad (\rho > B).$$

The reflected velocity at the singularity is then given by

$$(q_{0\phi})_{(B,0,0)} = -\frac{3}{\pi}\int_0^\infty [I_1(B\lambda)]^2 \frac{K_1(D\lambda)}{I_1(D\lambda)}\,d\lambda$$

and evidently is similar to that for a stokeslet moving parallel to a plane wall at distance $(D-B)$ namely, $3/[4(D-B)]$ (see Happel & Brenner 1973). Thus $(D-B)^{-1}$ is a measure of the accuracy of the values in table 1 when the flow occurs within a concentric cylindrical container of radius D.

4. Antisedimentation tube

The 'antisedimentation' tube (Dill & Brenner 1983; Nadim *et al.* 1985), described in I, is a device that permits a non-neutrally buoyant sphere to remain permanently suspended against the force of gravity by balancing its downward settling motion against an upward fluid current created by the steady rotation of a fluid-filled circular cylinder rotating about a horizontal axis. The simplified model considered in I neglects the wall effects and regards the sphere of radius a as fixed with its centre at distance b ($\gg a$) from the axis of rotation and in the same horizontal plane. Thus the outer field sees the fixed sphere as a point force of magnitude $6\pi\rho^* Ua$ which, after non-dimensionalization, is expressed by including the forcing term $-6\pi T^{\frac{1}{2}}\hat{y}\delta(x-B)\,\delta(y)\,\delta(z)$ on the right-hand side of (2.14), where T and B are defined by (2.17).

Hence, on writing $\boldsymbol{q} \sim T^{\frac{1}{2}}\boldsymbol{Q}(\boldsymbol{r})$, $P \sim T^{\frac{1}{2}}P_0(\boldsymbol{r})$ and including the point force term in (2.14), the perturbation field is, to leading order, governed by

$$\nabla^2\boldsymbol{Q}-\nabla P_0-\left[\frac{\partial Q_\rho}{\partial\phi}\hat{\boldsymbol{\rho}}+\frac{\partial Q_\phi}{\partial\phi}\hat{\boldsymbol{\phi}}+\frac{\partial Q_z}{\partial\phi}\hat{\boldsymbol{z}}\right]-2\hat{\boldsymbol{z}}\times\boldsymbol{Q} = 6\pi\hat{\boldsymbol{y}}\delta(\boldsymbol{r}-\hat{\boldsymbol{x}}B), \tag{4.1}$$

$$\nabla\cdot\boldsymbol{Q} = 0,$$

$$\boldsymbol{Q}\to\boldsymbol{0} \quad \text{as } |\boldsymbol{r}|\to\infty.$$

Here the Coriolis term has the usual factor 2, omitted in I, and there is a sign change in the linearized inertia term of Oseen type and the point force term owing to the reversal of the relative motion of the sphere and fluid. In making the necessary changes to the analysis of §4.2 of I, it is convenient to use $\mathrm{e}^{-\mathrm{i}m\phi}$ dependence whence the correct expressions for the velocity components and z-component of vorticity are found to be

$$Q_z = \frac{-3\mathrm{i}}{2\pi}\int_0^\infty \lambda\sin\lambda z\,\mathrm{d}\lambda \sum_{m=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}m\phi}\int_0^\infty \frac{\xi^2 J_m(\rho\xi)}{\Delta_m}\{(\lambda^2+\xi^2-\mathrm{i}m)\,[J_{m-1}(B\xi) + J_{m+1}(B\xi)]-2\mathrm{i}[J_{m-1}(B\xi)-J_{m+1}(B\xi)]\}\,\mathrm{d}\xi, \tag{4.2}$$

$$Q_\rho-\mathrm{i}Q_\phi = -\frac{3\mathrm{i}}{2\pi}\int_0^\infty \cos\lambda z\,\mathrm{d}\lambda \sum_{m=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}m\phi}\int_0^\infty \frac{\xi J_{m+1}(\rho\xi)}{\Delta_m}\{(\lambda^2+\xi^2-\mathrm{i}m)\,\xi^2[J_{m-1}(B\xi) - J_{m+1}(B\xi)]-2[\lambda^2+\xi^2-\mathrm{i}(m-2)]\,\lambda^2 J_{m+1}(B\xi)\}\,\mathrm{d}\xi, \tag{4.3}$$

$$\omega_z = \frac{3}{2\pi}\int_0^\infty \cos\lambda z\,\mathrm{d}\lambda \sum_{m=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}m\phi}\int_0^\infty \frac{\xi^2 J_m(\rho\xi)}{\Delta_m}\{(\lambda^2+\xi^2)\,(\lambda^2+\xi^2-\mathrm{i}m)\,[J_{m-1}(B\xi) - J_{m+1}(B\xi)]-2\mathrm{i}\lambda^2[J_{m-1}(B\xi)+J_{m+1}(B\xi)]\}\,\mathrm{d}\xi, \tag{4.4}$$

where
$$\Delta_m = (\lambda^2+\xi^2)\,(\lambda^2+\xi^2-\mathrm{i}m)^2+4\lambda^2 = \chi[(\chi-\mathrm{i}m)^2+4\eta^2], \tag{4.5}$$

with χ, η defined by (3.15). This denominator is indicative of the sixth-order equation (2.15).

Let $\boldsymbol{Q}^{(\mathrm{H})}$ and $\boldsymbol{Q}^{(\mathrm{s})}$ denote the inertialess and stokeslet solutions respectively, as in I. Then $(\omega_z^{(\mathrm{H})})_{(B,0,0)}$ and $(Q_\rho^{(\mathrm{s})})_{(B,0,0)}$ are evidently zero and, from (4.3) with use of the Appendix,

$$[Q_\rho^{(\mathrm{H})}-\mathrm{i}(Q_\phi^{(\mathrm{H})}-Q_\phi^{(\mathrm{s})})]_{(B,0,0)} = \frac{3\mathrm{i}}{4\pi}\int_0^\infty\int_0^1 \chi^{\frac{1}{2}}\,\mathrm{d}\chi\,\mathrm{d}\eta\left[\frac{\chi(1-\eta^2)}{\chi^2+4\eta^2}-\frac{1+\eta^2}{\chi}+\frac{2(\chi+2\mathrm{i})\,\eta^2}{\chi^2+4\eta^2}\right]$$

$$= \frac{3\mathrm{i}}{\pi}\int_0^\infty\int_0^1 \frac{\chi^{\frac{1}{2}}\eta^2}{\chi^2+4\eta^2}\left[-\frac{1+\eta^2}{\chi}+\mathrm{i}\right]\mathrm{d}\chi\,\mathrm{d}\eta.$$

Thus, as shown by Herron *et al.* (1975),

$$(Q_\rho^{(\mathrm{H})})_{(B,0,0)} = -\tfrac{3}{5}, \quad (Q_\phi^{(\mathrm{H})}-Q_\phi^{(\mathrm{s})})_{(B,0,0)} = \tfrac{5}{7},$$

and hence the dimensionless force $\boldsymbol{F}$ and torque $M\hat{\boldsymbol{z}}$ exerted on the sphere by the fluid are given by

$$\boldsymbol{F} = -6\pi\{\hat{\boldsymbol{y}}+T^{\frac{1}{2}}[-\tfrac{3}{5}\hat{\boldsymbol{x}}+\tfrac{5}{7}\hat{\boldsymbol{y}}+(Q_\rho-Q_\rho^{(\mathrm{H})})_{(B,0,0)}\,\hat{\boldsymbol{x}}+(Q_\phi-Q_\phi^{(\mathrm{H})})_{(B,0,0)}\,\hat{\boldsymbol{y}}]+O(T)\}, \tag{4.6}$$

$$M = -8\pi[1+\tfrac{1}{2}T^{\frac{1}{2}}(\omega_z)_{(B,0,0)}+O(T)]. \tag{4.7}$$

In evaluating (4.3) and (4.4) at $(B,0,0)$, the following formulae, analogous to (3.13), are required.

$$(\lambda^2+\xi^2)\sum_{m=-\infty}^{\infty} s_m\left(\frac{\lambda^2+\xi^2-im}{\Delta_m}-\frac{\lambda^2+\xi^2}{\Delta_0}\right) = \sum_{m=-\infty}^{\infty} s_m\frac{\chi-im}{(\chi-im)^2+4\eta^2}-s(0)\frac{\chi}{\chi^2+4\eta^2}$$
$$= \int_0^\infty [s(\psi)-s(0)]\,\mathrm{e}^{-\chi\psi}\cos 2\eta\psi\,\mathrm{d}\psi,$$

$$2\lambda(\lambda^2+\xi^2)^{\frac{1}{2}}\sum_{m=-\infty}^{\infty} s_m\left(\frac{1}{\Delta_m}-\frac{1}{\Delta_0}\right) = \sum_{m=-\infty}^{\infty} s_m\frac{2\eta}{(\chi-im)^2+4\eta^2}-s(0)\frac{2\eta}{\chi^2+4\eta^2}$$
$$= \int_0^\infty [s(\psi)-s(0)]\,\mathrm{e}^{-\chi\psi}\sin 2\eta\psi\,\mathrm{d}\psi.$$

Thus, on using the summations listed in the Appendix, it follows from (4.3) and (4.4) that

$$(Q_\rho-\mathrm{i}Q_\phi-Q_\rho^{(\mathrm{H})}+\mathrm{i}Q_\phi^{(\mathrm{H})})_{(B,0,0)} = \frac{3\mathrm{i}}{4\pi}\int_0^\infty\int_0^1 \chi^{\frac{1}{2}}\,\mathrm{d}\chi\,\mathrm{d}\eta\left\{\int_0^\infty \mathrm{e}^{-\chi\psi}\cos 2\eta\psi\right.$$
$$\times[(1-\eta^2)\,J_2[2B\chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\sin\tfrac{1}{2}\psi]+(1+\eta^2)\{J_0[2B\chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\sin\tfrac{1}{2}\psi]\,\mathrm{e}^{-\mathrm{i}\psi}-1\}]\,\mathrm{d}\psi$$
$$\left.+2\mathrm{i}\eta\int_0^\infty \mathrm{e}^{-\chi\psi}\sin 2\eta\psi\{J_0[2B\chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}\sin\tfrac{1}{2}\psi]\,\mathrm{e}^{-\mathrm{i}\psi}-1\}\,\mathrm{d}\psi\right\}$$

$$\tfrac{1}{2}(\omega_z)_{(B,0,0)} = -\frac{3}{8\pi}\int_0^\infty\int_0^1 \chi(1-\eta^2)^{\frac{1}{2}}\,\mathrm{d}\chi\,\mathrm{d}\eta\int_0^\infty \mathrm{e}^{-\chi\psi}\frac{J_1[2B\chi^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}|\sin\frac{1}{2}\psi|]}{|\sin\frac{1}{2}\psi|}$$
$$\times\{(1-\cos\psi)\cos 2\eta\psi-\eta\sin\psi\sin 2\eta\psi\}\,\mathrm{d}\psi.$$

The η-integrations involving Bessel functions can be derived from the result

$$\int_0^1 J_0[\alpha(1-\eta^2)^{\frac{1}{2}}]\cos 2\eta\psi\,\mathrm{d}\eta = \frac{\sin[(\alpha^2+4\psi^2)^{\frac{1}{2}}]}{(\alpha^2+4\psi^2)^{\frac{1}{2}}},$$

and the others correspond to the simpler case $\alpha = 0$. Hence

$$(Q_\rho-Q_\rho^{(\mathrm{H})})_{(B,0,0)}+\mathrm{i}(Q_\phi-Q_\phi^{(\mathrm{H})})_{(B,0,0)} = -\frac{3\mathrm{i}}{4\pi}\int_0^\infty\int_0^\infty \chi^{\frac{1}{2}}\mathrm{e}^{-\chi\psi}\,\mathrm{d}\chi\,\mathrm{d}\psi$$
$$\times\left\{\left(\frac{\sin 2v}{2v}-\cos 2v\right)\left[\frac{3}{4v^2}\left(1-\frac{\psi^2}{v^2}\right)(1+\mathrm{e}^{\mathrm{i}\psi})-\frac{(\frac{1}{2}+\mathrm{i}\psi)}{v^2}\mathrm{e}^{\mathrm{i}\psi}\right]\right.$$
$$\left.-\frac{\sin 2v}{2v}\left[1-\mathrm{e}^{\mathrm{i}\psi}-\frac{\psi^2}{v^2}(1+\mathrm{e}^{\mathrm{i}\psi})\right]-\frac{\sin 2\psi}{\psi}+\left(\frac{\sin 2\psi}{2\psi}-\cos 2\psi\right)\left(\frac{1}{2\psi^2}+\frac{\mathrm{i}}{\psi}\right)\right\} \tag{4.8}$$

$$\tfrac{1}{2}(\omega_z)_{(B,0,0)} = -\frac{3B}{16\pi}\int_0^\infty\int_0^\infty \chi^{\frac{3}{2}}\mathrm{e}^{-\chi\psi}v^{-2}\,\mathrm{d}\chi\,\mathrm{d}\psi$$
$$\times\left\{\frac{\psi}{v}\sin\psi\sin 2v+\left(\frac{\sin 2v}{2v}-\cos 2v\right)\left(1-\cos\psi-\frac{3\psi}{2v^2}\sin\psi\right)\right\}, \tag{4.9}$$

where $$v^2 = \psi^2+B^2\chi\sin^2\tfrac{1}{2}\psi.$$

If the χ-integrations are performed first, then convergence will depend on the behaviour of the outer integrands as $\psi\to 0$. In this limit, it may be shown that { } and

$v^{-2}\{\ \}$ in the respective double integrals are asymptotic to $\frac{4}{3}\mathrm{i}\psi$ and $\frac{2}{15}\psi^2$ which are independent of v and hence χ. Thus the outer integrands are respectively asymptotic to $1/2(\pi\psi)^{\frac{1}{2}}$ and $-3B/160(\pi\psi)^{\frac{1}{2}}$ as $\psi \to 0$ and convergence is assured.

Values of the integrals in (4.8), (4.9) have been computed numerically in order to tabulate in table 2 the B-dependent force and torque coefficients in the formulae (2.18) where (cf. (4.6), (4.7))

$$C_x = \tfrac{3}{5} - (Q_\rho - Q_\rho^{(\mathrm{H})})_{(B,0,0)}, \quad C_y = \tfrac{5}{7} + (Q_\phi - Q_\phi^{(\mathrm{H})})_{(B,0,0)}, \quad C_z = -\tfrac{1}{2}(\omega_z)_{(B,0,0)} \tag{4.10}$$

An estimate of the error due to the container of radius $D \gg B$ is precluded by the sixth-order denominator in (4.5).

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Appendix

The following summations, derived from formula 8.530.2 of Gradshteyn & Ryzhik (1980), are of the type $s(\phi)$ in (3.7)

$$\sum_{m=-\infty}^{\infty} J_{m+1}(\rho\xi)\, J_{m+1}(B\xi)\, \mathrm{e}^{\mathrm{i}m\phi} = J_0[\xi(\rho^2+B^2-2\rho B\cos\phi)^{\frac{1}{2}}]\,\mathrm{e}^{-\mathrm{i}\phi},$$

$$\sum_{m=-\infty}^{\infty} J_{m+1}(\rho\xi)\, J_{m-1}(B\xi)\, \mathrm{e}^{\mathrm{i}m\phi} = \left(\frac{\rho\,\mathrm{e}^{\mathrm{i}\phi}-B}{\rho-B\,\mathrm{e}^{\mathrm{i}\phi}}\right) J_2[\xi(\rho^2+B^2-2\rho B\cos\phi)^{\frac{1}{2}}],$$

$$\sum_{m=-\infty}^{\infty} J_m(\rho\xi)\,[J_{m-1}(B\xi) \pm J_{m+1}(B\xi)]\,\mathrm{e}^{\mathrm{i}m\phi} = \begin{pmatrix} \mathrm{i}\rho\sin\phi \\ \rho\cos\phi - B \end{pmatrix} \frac{2J_1[\xi(\rho^2+B^2-2\rho B\cos\phi)^{\frac{1}{2}}]}{(\rho^2+B^2-2\rho B\cos\phi)^{\frac{1}{2}}}.$$

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